Application of Kamal Transform for Solving Linear Volterra Integral Equations of First Kind

Sudhanshu Aggarwal[1], Nidhi Sharma[2], Raman Chauhan[3]
[1]Assistant Professor, Department of Mathematics, National P.G. College Barhalganj, Gorakhpur-273402, U.P., India
[2,3]Assistant Professor, Department of Mathematics, Noida Institute of Engineering & Technology, Greater Noida-201306, U.P., India
sudhanshu30187@gmail.com, 6nidhisharma@gmail.com, raman28chauhan@gmail.com

Abstract: Many advance problems of biology, chemistry, physics and engineering can represent mathematically in the form of Volterra integral equations of first kind. In this paper, we used Kamal transform for solving linear Volterra integral equations of first kind and some applications are given in order to demonstrate the effectiveness of Kamal transform for solving linear Volterra integral equations of first kind.

Keywords: Linear Volterra integral equation of first kind, Kamal transform, Convolution theorem, Inverse Kamal transform.

1. INTRODUCTION

The linear Volterra integral equation of first kind is given by [1-12]

$$f(x) = \int_0^x k(x,t)u(t)dt \text{ .... } (1)$$

where the unknown function $u(x)$, that will be determined, occurs only inside the integral sign. The kernel $k(x,t)$and the function $f(x)$ are given real-valued functions.

The Kamal transform of the function $F(t)$ is defined as [16]:

$$K\{F(t)\} = \int_0^\infty F(t)e^{\frac{-t}{v}}dt$$

$$= G(v), t \geq 0, k_1 \leq v \leq k_2$$

where K is Kamal transform operator.

The Kamal transform of the function $F(t)$ for $t \geq 0$ exist if $F(t)$ is piecewise continuous and of exponential order. These conditions are only sufficient conditions for the existence of Kamal transform of the function $F(t)$.

Abdelilah and Hassan [17] applied Kamal transform for solving partial differential equations. Fadhil [18] discussed the convolution for Kamal and Mahgoub transforms. Taha et. al. [19] defined the dualities between Kamal & Mahgoub integral transforms and some famous integral transforms. Aggarwal et al. [20] discussed a new application of Kamal transform for solving linear Volterra integral equations. Gupta et al. [21] gave the solution of linear partial integro-differential equations using Kamal transform. Aggarwal [22] defined Kamal transform of Bessel's functions. Numerical solution for Volterra integral equations of the first kind via Quadrature rule was given by Mirzaee [23]. Maleknejad et al. [24] gave the numerical solution of Volterra integral equations of first kind by using a recursive scheme. Babolian and Masouri [25] applied direct method to solve Volterra integral equation of first kind using operational matrix with block-pulse functions.

The aim of this work is to establish exact solutions for linear Volterra integral equation of first kind using Kamal transform without large computational work.

2. Linearity property of Kamal transforms [22]:

If $K\{F(t)\} = H(v)$ and $K\{G(t)\} = I(v)$ then $K\{aF(t) + bG(t)\} = aK\{F(t)\} + bK\{G(t)\}$

$\Rightarrow K\{aF(t) + bG(t)\} = aH(v) + bI(v),$

where a, b are arbitrary constants.

3. Kamal transform of some elementary functions [20-22]:

S.N.	$F(t)$	$K\{F(t)\} = G(v)$
1.	1	v
2.	t	v^2
3.	t^2	$2!\, v^3$
4.	$t^n, n\epsilon N$	$n!\, v^{n+1}$
5.	$t^n, n > -1$	$\Gamma(n+1)v^{n+1}$
6.	e^{at}	$\frac{v}{1-av}$
7.	$sinat$	$\frac{av^2}{1+a^2v^2}$
8.	$cosat$	$\frac{v}{1+a^2v^2}$
9.	$sinhat$	$\frac{av^2}{1-a^2v^2}$
10.	$coshat$	$\frac{v}{1-a^2v^2}$

4. Convolution of two functions [13-15]:

Convolution of two functions $F(t)$ and $G(t)$ is denoted by $F(t) * G(t)$ and it is defined by

$$F(t) * G(t) = F * G = \int_O^t F(x)G(t-x)dx$$

$$= \int_O^t F(t-x)G(x)dx$$

5. Convolution theorem for Kamal transforms [18, 20-22]:

If $K\{F(t)\} = H(v)$ and $K\{G(t)\} = I(v)$ then $K\{F(t) * G(t)\} = K\{F(t)\}K\{G(t)\} = H(v)I(v)$

6. Inverse Kamal transform [20-22]:

If $K\{F(t)\} = G(v)$ then $F(t)$ is called the inverse Kamal transform of $G(v)$ and mathematically it is defined as

$$F(t) = K^{-1}\{G(v)\}$$

where K^{-1} is the inverse Kamal transform operator.

7. Inverse Kamal transform of some elementary functions [20-22]:

S.N.	$G(v)$	$F(t) = K^{-1}\{G(v)\}$
1.	v	1
2.	v^2	t
3.	v^3	$\frac{t^2}{2!}$
4.	$v^{n+1}, n\epsilon N$	$\frac{t^n}{n!}$
5.	$v^{n+1}, n > -1$	$\frac{t^n}{\Gamma(n+1)}$
6.	$\frac{v}{1-av}$	e^{at}
7.	$\frac{v^2}{1+a^2v^2}$	$\frac{sinat}{a}$

8.	$\frac{v}{1+a^2v^2}$	$cosat$
9.	$\frac{v^2}{1-a^2v^2}$	$\frac{sinhat}{a}$
10.	$\frac{v}{1-a^2v^2}$	$coshat$

8. Kamal transform of Bessel's functions [22]:

a) Kamal transform of Bessel's function of zero order $J_0(t)$:

$$K\{J_0(t)\} = \frac{v}{\sqrt{(1+v^2)}}$$

b) Kamal transform of Bessel's function of order one $J_1(t)$:

$$K\{J_1(t)\} = 1 - \frac{1}{\sqrt{(1+v^2)}}$$

9. Kamal transforms for linear Volterra integral equations of first kind:

In this work we will assume that the kernel $k(x,t)$of (1) is a difference kernel that can be expressed by the difference$(x-t)$. The linear Volterra integral equation of first kind (1) can thus be expressed as

$$f(x) = \int_0^x k(x-t)u(t)dt \ldots\ldots\ldots\ldots\ldots (2)$$

Applying the Kamal transform to both sides of(2), we have

$$K\{f(x)\} = K\{\int_0^x k(x-t)u(t)dt\} \ldots\ldots (3)$$

Using convolution theorem of Kamal transform, we have

$$K\{f(x)\} = K\{k(x)\}K\{u(x)\}$$

$$\Rightarrow K\{u(x)\} = \left[\frac{K\{f(x)\}}{K\{k(x)\}}\right] \ldots\ldots\ldots\ldots (4)$$

Operating inverse Kamal transform on both sides of(4), we have

$$u(x) = K^{-1}\left\{\left[\frac{K\{f(x)\}}{K\{k(x)\}}\right]\right\} \ldots\ldots\ldots\ldots (5)$$

which is the required solution of (2).

10. Applications:

In this section, some applications are given in order to demonstrate the effectiveness of Kamal transform for solving linear Volterra integral equations of first kind.

A. **Application:1** Consider linear Volterra integral equation of first kind

$$x = \int_0^x e^{(x-t)} u(t)dt \ldots\ldots\ldots\ldots (6)$$

Applying the Kamal transform to both sides of(6), we have

$$K\{x\} = K\left\{\int_0^x e^{(x-t)} u(t)dt\right\} \ldots\ldots\ldots\ldots (7)$$

Using convolution theorem of Kamal transform on (7), we have

$$v^2 = K\{e^x\}K\{u(x)\}$$

$$\Rightarrow v^2 = \left[\frac{v}{1-v}\right] K\{u(x)\}$$

$$\Rightarrow K\{u(x)\} = v - v^2 \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Operating inverse Kamal transform on both sides of(8), we have

$$u(x) = K^{-1}\{v - v^2\} = K^{-1}\{v\} - K^{-1}\{\, v^2 \,\}$$

$$\Rightarrow u(x) = 1 - x \ldots\ldots\ldots\ldots\ldots (9)$$

which is the required exact solution of (6).

B. **Application:2** Consider linear Volterra integral equation of first kind

$$sinx = \int_0^x e^{(x-t)}u(t)\, dt \ldots\ldots (10)$$

Applying the Kamal transform to both sides of(10), we have

$$K\{sinx\} = K\{\int_0^x e^{(x-t)} u(t)\, dt\} \ldots\ldots\ldots\ldots (11)$$

Using convolution theorem of Kamal transform on(11), we have

$$\frac{v^2}{1+v^2} = K\{e^x\}K\{u(x)\}$$

$$\Rightarrow \frac{v^2}{1+v^2} = \left[\frac{v}{1-v}\right] K\{u(x)\}$$

$$\Rightarrow K\{u(x)\} = \frac{v(1-v)}{1+v^2} = \frac{v}{1+v^2} - \frac{v^2}{1+v^2} \ldots\ldots (12)$$

Operating inverse Kamal transform on both sides of(12), we have

$$u(x) = K^{-1}\left\{\frac{v}{1+v^2}\right\} - K^{-1}\left\{\frac{v^2}{1+v^2}\right\}$$

$$\Rightarrow\ u(x) = cosx - sinx \ldots\ldots\ldots\ldots\ldots (13)$$

which is the required exact solution of (10).

C. Application:3 Consider linear Volterra integral equation of first kind

$$sinx = \int_0^x J_0(x-t)u(t)\, dt \ldots\ldots (14)$$

Applying the Kamal transform to both sides of(14), we have

$$K\{sinx\} = K\{\int_0^x J_0(x-t)u(t)\, dt\} \ldots (15)$$

Using convolution theorem of Kamal transform on(15), we have

$$\frac{v^2}{1+v^2} = K\{J_0(x)\}K\{u(x)\}$$

$$\Rightarrow \frac{v^2}{1+v^2} = \left[\frac{v}{\sqrt{(1+v^2)}}\right] K\{u(x)\}$$

$$\Rightarrow K\{u(x)\} = \frac{v}{\sqrt{(1+v^2)}} \ldots\ldots\ldots\ldots (16)$$

Operating inverse Kamal transform on both sides of(16), we have

$$u(x) = K^{-1}\left\{\frac{v}{\sqrt{(1+v^2)}}\right\} = J_0(x) \ldots\ldots\ldots (17)$$

which is the required exact solution of (14).

D. Application:4 Consider linear Volterra integral equation of first kind

$$x^2 = \frac{1}{2}\int_0^x (x-t)u(t)\, dt \ldots\ldots (18)$$

Applying the Kamal transform to both sides of(18), we have

$$K\{x^2\} = \frac{1}{2}K\{\int_0^x (x-t)u(t)\, dt\} \ldots (19)$$

Using convolution theorem of Kamal transform on(19), we have

$$2!\, v^3 = \frac{1}{2}.K\{x\}K\{u(x)\}$$

$$\Rightarrow 2!\, v^3 = \frac{1}{2}.\left[v^2\right]K\{u(x)\}$$

$$\Rightarrow K\{u(x)\} = 4v \ldots\ldots\ldots\ldots (20)$$

Operating inverse Kamal transform on both sides of(20), we have

$$u(x) = 4K^{-1}\{v\} = 4 \ldots\ldots\ldots (21)$$

which is the required exact solution of (18).

E. Application:5 Consider linear Volterra integral equation of first kind

$$x = \int_0^x e^{-(x-t)}\, u(t)dt \ldots\ldots (22)$$

Applying the Kamal transform to both sides of (22), we have

$$K\{x\} = K\{\int_0^x e^{-(x-t)}\, u(t)dt\} \ldots (23)$$

Using convolution theorem of Kamal transform on(23), we have

$$v^2 = K\{e^{-x}\}K\{u(x)\}$$

$$\Rightarrow v^2 = \left[\frac{v}{v+1}\right] K\{u(x)\}$$

$$\Rightarrow K\{u(x)\} = v^2 + v \ldots\ldots\ldots\ldots (24)$$

Operating inverse Kamal transform on both sides of(24), we have

$$u(x) = K^{-1}\{v^2 + v\} = K^{-1}\{v^2\} + K^{-1}\{\, v\,\}$$

$$\Rightarrow u(x) = x + 1 \ldots\ldots\ldots\ldots\ldots (25)$$

which is the required exact solution of (22).

F. Application:6 Consider linear Volterra integral equation of first kind

$$sinx = \int_0^x u(x-t)\, u(t) dt \ldots\ldots(26)$$

Applying the Kamal transform to both sides of (26), we have

$$K\{sinx\} = K\left\{\int_0^x u(x-t)\, u(t) dt\right\} \ldots. (27)$$

Using convolution theorem of Kamal transform on(27), we have

$$\frac{v^2}{1+v^2} = K\{u(x)\}K\{u(x)\}$$

$$\Rightarrow [K\{u(x)\}]^2 = \frac{v^2}{v^2+1}$$

$$\Rightarrow K\{u(x)\} = \pm \frac{v}{\sqrt{v^2+1}} \ldots\ldots\ldots\ldots (28)$$

Operating inverse Kamal transform on both sides of(28), we have

$$u(x) = \pm K^{-1}\left\{\frac{v}{\sqrt{v^2+1}}\right\}$$

$$\Rightarrow u(x) = \pm J_0(x) \ldots\ldots\ldots\ldots (29)$$

which is the required exact solution of (26).

G. Application:7 Consider linear Volterra integral equation of first kind

$$x = \int_0^x u(t)\, dt \ldots\ldots(30)$$

Applying the Kamal transform to both sides of (30), we have

$$K\{x\} = K\left\{\int_0^x u(t)\, dt\right\} \ldots. (31)$$

Using convolution theorem of Kamal transform on(31), we have

$$v^2 = K\{1\}K\{u(x)\}$$

$$\Rightarrow v^2 = v.K\{u(x)\}$$

$$\Rightarrow K\{u(x)\} = v \ldots\ldots\ldots\ldots (32)$$

Operating inverse Kamal transform on both sides of(32), we have

$$u(x) = K^{-1}\{v\} = 1 \ldots\ldots\ldots\ldots (33)$$

which is the required exact solution of (30).

H. Application:8 Consider linear Volterra integral equation of first kind

$$1 - J_0(x) = \int_0^x u(t)\, dt \ldots\ldots(34)$$

Applying the Kamal transform to both sides of (34), we have

$$K\{1\} - K\{J_0(x)\} = K\left\{\int_0^x u(t)\, dt\right\} \ldots. (35)$$

Using convolution theorem of Kamal transform on(35), we have

$$v - \frac{v}{\sqrt{v^2+1}} = K\{1\}K\{u(x)\}$$

$$\Rightarrow v - \frac{v}{\sqrt{v^2+1}} = v.K\{u(x)\}$$

$$\Rightarrow K\{u(x)\} = 1 - \frac{1}{\sqrt{(1+v^2)}} \ldots\ldots\ldots\ldots (36)$$

Operating inverse Kamal transform on both sides of(36), we have

$$u(x) = K^{-1}\left\{1 - \frac{1}{\sqrt{(1+v^2)}}\right\} = J_1(x) \ldots\ldots(37)$$

which is the required exact solution of (34).

I. Application:9 Consider linear Volterra integral equation of first kind

$$J_0(x) - cosx = \int_0^x J_0(x-t)u(t)\, dt \ldots(38)$$

Applying the Kamal transform to both sides of (38), we have

$K\{J_0(x)\} - K\{cosx\} = K\{\int_0^x J_0(x-t)u(t)\,dt\}$..(39)

Using convolution theorem of Kamal transform on(39), we have

$$\frac{v}{\sqrt{v^2+1}} - \frac{v}{v^2+1} = K\{J_0(x)\}K\{u(x)\}$$

$$\Rightarrow \frac{v}{\sqrt{v^2+1}} - \frac{v}{v^2+1} = \frac{v}{\sqrt{(1+v^2)}}.K\{u(x)\}$$

$$\Rightarrow K\{u(x)\} = 1 - \frac{1}{\sqrt{(1+v^2)}} \ldots\ldots\ldots\ldots (40)$$

Operating inverse Kamal transform on both sides of(40), we have

$$u(x) = K^{-1}\left\{1 - \frac{1}{\sqrt{(1+v^2)}}\right\} = J_1(x)\ldots\ldots(41)$$

which is the required exact solution of (38).

J. Application:10 Consider linear Volterra integral equation of first kind

$cosx - J_0(x) = -\int_0^x J_1(x-t)u(t)\,dt$…..(42)

Applying the Kamal transform to both sides of (42), we have

$$K\{cosx\} - K\{J_0(x)\} = -K\int_0^x J_1(x-t)u(t)\,dt \quad \ldots\ldots\ldots\ldots(43)$$

Using convolution theorem of Kamal transform on(43), we have

$$\frac{v}{v^2+1} - \frac{v}{\sqrt{(1+v^2)}} = -K\{J_1(x)\}K\{u(x)\}$$

$$\Rightarrow \frac{v}{v^2+1} - \frac{v}{\sqrt{(1+v^2)}} = -\left[1 - \frac{1}{\sqrt{(1+v^2)}}\right].K\{u(x)\}$$

$$\Rightarrow K\{u(x)\} = \frac{v}{\sqrt{(1+v^2)}} \ldots\ldots\ldots\ldots (44)$$

Operating inverse Kamal transform on both sides of(44), we have

$$u(x) = K^{-1}\left\{\frac{v}{\sqrt{(1+v^2)}}\right\} = J_0(x)\ldots\ldots\ldots\ldots (45)$$

which is the required exact solution of (42).

11. CONCLUSION

In this paper, we have successfully developed the Kamal transform for solving linear Volterra integral equations of first kind. The given applications showed that the exact solution have been obtained using very less computational work and spending a very little time. The proposed scheme can be applied for other linear Volterra integral equations and their system.

REFERENCES

[1] A.C. Pipkin. “A course on integral equations”. Springer-Verlag, New York, 1991.
[2] H. Hochstadt. “Integral equations”. Wiley, New York, 1973.
[3] R.P. Kanwal. “Linear integral equations”. Academic Press, New York, 1971.
[4] W.V. Lovitt. “Linear integral equations”. Dover, New York, 1950.
[5] F.G. Tricomi. “Integral equations”. Interscience, New York, 1957.
[6] A.M. Wazwaz. “A first course in integral equations”. World Scientific, Singapore, 1997.
[7] A.V. Bitsadze. “Integral equations of first kind”. World Scientific, Singapore.

[8] A.D. Polyanin and A.V. Manzhirov. "Handbook of integral equations". Chapman & Hall/CRC, 2008.
[9] B.L. Moiseiwitsch. "Integral equations". Dover Publications, New York, 2005.
[10] W. Hackbusch. "Integral equations: Theory and numerical treatment". Birkhauser Verlag, Berlin, 1995.
[11] P. Linz. "Analytical and numerical methods for Volterra equations". SIAM, Philadelphia, 1984.
[12] A. J. Jerri. "Introduction to integral equations with applications". Wiley, New York, 1999.
[13] M.D. Raisinghania. "Advanced differential equations". S.Chand & Company, New-Delhi, 2015.
[14] M.D. Raisinghania. "Integral equations and boundary value problems". S.Chand & Company, New-Delhi, 2017.
[15]Lokenath Debnath and D. Bhatta. "Integral transforms and their applications". Second edition, Chapman & Hall/CRC, 2006.
[16] K. Abdelilah and S. Hassan. "The new integral transform Kamal Transform". Advances in theoretical and applied mathematics, 11(4), 451-458, 2016.
[17] K. Abdelilah and S. Hassan. "The use of Kamal transform for solving partial differential equations". Advances in theoretical and applied mathematics, 12(1), 7-13, 2017 .
[18] R.A. Fadhil. "Convolution for Kamal and Mahgoub transforms". Bulletin of mathematics and statistics research, 5(4), 11-16, 2017.
[19] N.E.H. Taha, R.I. Nuruddeen, K. Abdelilah and S. Hassan. "Dualities between Kamal & Mahgoub Integral Transforms and Some Famous Integral Transforms". British Journal of Applied Science & Technology, 20(3), 1-8, 2017.
[20] S. Aggarwal, R. Chauhan and N. Sharma. "A new application of Kamal transform for solving linear Volterra integral equations". International Journal of Latest Technology in Engineering, Management & Applied Science, 7(4), 138-140, 2018.
[21] A.R. Gupta, S. Aggarwal and D. Agrawal. "Solution of linear partial integro-differential equations using Kamal transform". International Journal of Latest Technology in Engineering, Management & Applied Science, 7(7), 88-91, 2018.
[22] S. Aggarwal. "Kamal transform of Bessel's functions". International Journal of Research and Innovation in Applied Science, 3(7), 1-4, 2018.
[23] Farshid Mirzaee. "Numerical solution for Volterra integral equations of the first kind via Quadrature rule". Applied Mathematical Sciences, 6(20), 969-974, 2012.
[24] K. Maleknejad, M. Roodaki and H. Almasieh. "Numerical solution of Volterra integral equations of first kind by using a recursive scheme". Journal of Mathematical Extension, 3(2), 113-121, 2009.
[25] E. Babolian and Z. Masouri. "Direct method to solve Volterra integral equation of first kind using operational matrix with block-pulse functions". Journal of Computational and Applied Mathematics, 220, 51-57, 2008.

Sudhanshu Aggarwal received his M.Sc. degree from M.S. College, Saharanpur in 2007. He has also qualified CSIR NET examination (June-2010, June-2012, June-2013, June-2014 & June-2015) in Mathematical Sciences. He is working as an Assistant Professor in National P.G. College Barhalganj Gorakhpur. He is equipped with an extraordinary caliber and appreciable academic potency. He has around ten years of teaching experience at various engineering colleges affiliated to AKTU. His fields of interest include Integral Transform Methods, Differential and Partial Differential Equations, Integral Equations and Number Theory.

Nidhi Sharma received her M.Sc. degree from Banasthali University Rajasthan in 2015. She is working as an Assistant Professor in Noida Institute of Engineering & Technology, Greater Noida since 2015. She has also qualified IIT Jam exam (2013). She has published many research papers in national and international journals. Her fields of interest include Optimization, Fuzzy Logic and Topological Spaces.

Raman Chauhan has completed M.Sc. (Mathematics) from Indian Institute of Technology, Roorkee in 2011. He is working at NIET, Greater Noida since 2013 as a dynamic, successful and responsible faculty member in the department of Mathematics. He has also qualified IIT Jam exam (2009) and Gate exam (2011, 2013, 2014 & 2015). His fields of interest include Optimization, Integral Equations and Abstract Algebra.